A SURGERY FORMULA FOR THE ASYMPTOTICS OF HIGHER DIMENSIONAL REIDEMEISTER TORSION AND SEIFERT FIBERED SPACES

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Abstract. We give a surgery formula for the asymptotic behavior of the sequence given by the logarithm of higher dimensional Reidemeister torsion. Applying the resulting formula to Seifert fibered spaces, we show that the growths of the sequences have the same order as the indices and give the explicit values for the limits of the leading coefficients. There are finitely many possibilities as the limit of the leading coefficient for a Seifert fibered space, which are assigned to each component in the representation spaces, and we also show the maximum is given by $-\chi \log 2$ where χ is the Euler characteristic of the base orbifold for a Seifert fibered space. The limit of the leading coefficient also gives a locally constant function on character varieties. This function takes the maximal value only on top-dimensional components for the SU(2)-character varieties of Seifert fibered homology spheres.

1. Introduction

This paper is devoted to the study of the asymptotics of higher dimensional Reidemeister torsion for *Seifert fibered spaces*. The higher dimensional Reidemeister torsion is defined for a 3-manifold and a sequence of homomorphisms from the fundamental group, which is given by the composition of an $\mathrm{SL}_2(\mathbb{C})$-*representation* of the fundamental group with all n-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$. It is of interest to observe the asymptotic behavior of higher dimensional Reidemeister torsion on the index n.

This study is motivated by the works of W. Müller [Mül] and P. Menal-Ferrer and J. Porti [MFPa], which revealed the relationship between the asymptotic behaviors of Ray–Singer and Reidemeister torsions for a hyperbolic 3-manifold and the hyperbolic volume. Their invariants are also defined by a sequence of $\mathrm{SL}_n(\mathbb{C})$-representations induced from the holonomy representation corresponding to the complete hyperbolic structure. The sequences of Ray–Singer and Reidemeister torsions have exponential growth and the logarithms have the order of n^2. They showed that the leading coefficient of the logarithm converges to the product of the hyperbolic volume and $-1/(4\pi)$.

We can also consider the same construction of Reidemeister torsion for non–hyperbolic 3-manifolds, especially for Seifert fibered spaces and the sequences given by the logarithm of the Reidemeister torsion for $\mathrm{SL}_n(\mathbb{C})$-representations. From the results of Müller and Menal-Ferrer and Porti, it is expected that the growth of the logarithm of higher dimensional Reidemeister torsion for a Seifert fibered space has the order of smaller than n^2 and the leading coefficient converges to some geometric quantity of the Seifert fibered space. Actually we will see that the order of growth is the same as n and the limits of the leading

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coefficient form a finite set, in which the maximum is given by $-\chi \log 2$ where χ is the Euler characteristic of the base orbifold of a Seifert fibered space.

To observe the asymptotic behaviors for Seifert fibered spaces, we establish a surgery formula for gluing solid tori to a compact orientable 3-manifold with torus boundary, since every Seifert fibered space admits the canonical decomposition into the trivial S^1-bundle over a compact surface and several solid tori (for details, see Subsection 4.1).

There is a problem which we have to consider on the choice of homomorphisms from fundamental groups into $\mathrm{SL}_2(\mathbb{C})$ since Seifert fibered spaces do not admit complete hyperbolic structures. Concerning this problem, we focus on the properties for the induced twisted chain complexes defined by $\mathrm{SL}_n(\mathbb{C})$-representations for hyperbolic 3-manifolds. It was shown in [Rag65, MFPb] that for a hyperbolic 3-manifold, the induced $\mathrm{SL}_{2N}(\mathbb{C})$-representations from the holonomy define *acyclic* chain complexes, i.e., all of those homology groups vanish. We will restrict our attention to $\mathrm{SL}_2(\mathbb{C})$-representations such that all the induced $\mathrm{SL}_{2N}(\mathbb{C})$-representations define the acyclic twisted chain complexes.

The following is a surgery formula (Theorem 3.6) for the limits of sequences given by the logarithm of higher dimensional Reidemeister torsion under some acyclicity conditions (Definition 3.4).

Surgery formula for the asymptotics (Theorem 3.6)**.** *Let $M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$ be a closed orientable 3-manifold obtained by gluing solid tori $S_1, \ldots, S_m$ with the slopes $\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}$ to a compact orientable manifold M with torus boundary.*

We denote by ρ an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M)$ which can be extended to a homomorphism of $\pi_1(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}))$ and by ρ_{2N} the induced $2N$-dimensional representation from ρ. Then the asymptotics of $\log|\mathrm{Tor}(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|$ is expressed as follows:

$$\text{(i)}\quad \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{(2N)^2} = \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M; \rho_{2N})|}{(2N)^2},$$

$$\text{(ii)}\quad \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{2N} = \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M; \rho_{2N})|}{2N} - \log 2 \sum_{j=1}^{m} \frac{1}{\lambda_j}$$

where $2\lambda_j$ is the order of $\rho(\ell_j)$ and ℓ_j is the homotopy class of the core in S_j.

Applying the above surgery formula to the decomposition of Seifert fibered spaces, we can show the asymptotic behaviors for Seifert fibered spaces.

Asymptotic behaviors for Seifert fibered spaces (Theorem 4.4 and Corollary 4.7)**.** *Let $M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$ denote a Seifert fibered space with the Seifert index:*

$$\{b, (o, g); (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}.$$

Then we can express the asymptotic behavior of $\log|\mathrm{Tor}(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|$ as follows:

$$\text{(i)}\quad \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{(2N)^2} = 0,$$

$$\text{(ii)}\quad \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{2N} = -\Big(2 - 2g - \sum_{j=1}^{m} \frac{\lambda_j - 1}{\lambda_j}\Big) \log 2$$

where $2\lambda_j$ is the order of $\rho(\ell_j)$ for the exceptional fiber ℓ_j. In particular, the second equality can be written as

$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{2N} = -\Big(2-2g-\sum_{j=1}^{m} \frac{\alpha_j - 1}{\alpha_j}\Big) \log 2 - \Big(\sum_{j=1}^{m} \Big(\frac{1}{\lambda_j} - \frac{1}{\alpha_j}\Big)\Big) \log 2.$$

The first term in the right hand side is equal to $-\chi \log 2$ where χ is the Euler characteristic of the base orbifold. Since each λ_j is a divisor of α_j, we can see that the maximum of the above limit is given by $-\chi \log 2$. Note that we only require that the original $\mathrm{SL}_2(\mathbb{C})$-representation sends a regular fiber to $-I$ for a Seifert fibered space.

Moreover the limit of the leading coefficient is determined by each component in the representation spaces. By the invariance of Reidemeister torsion under the conjugation of representations, we can assign each component of the character variety to the limit of the leading coefficient, namely, we can define a locally constant function on the character variety for a Seifert fibered space. We will discuss on which components our locally constant function takes the maximum and minimum for the SU(2)-character varieties of Seifert fibered homology 3-spheres in detail.

Organization. We review the definition of Reidemeister torsion and the construction of higher dimensional ones in Section 2. Section 3 is devoted to establish our surgery formula under the acyclicity conditions which are deduced from the observation in Subsection 3.1. The examples of the surgery formula for integral surgeries along torus knots are exhibited in Subsection 3.3. We discuss the asymptotic behaviors of the sequences given by the logarithm of higher dimensional Reidemeister torsion for Seifert fibered spaces in Section 4. We review on Seifert fibered spaces and prepare notations in Subsection 4.1. Subsection 4.2 gives a general formula of the asymptotic behavior for a Seifert fibered space. Furthermore we observe the relation between the limits of the leading coefficients and components in the SU(2)-character varieties for Seifert fibered homology 3-spheres in Subsection 4.3. The last Subsection 4.4 gives the explicit examples of limits of the leading coefficient and the SU(2)-character varieties.

2. Preliminaries

2.1. **Reidemeister torsion.**

Torsion for acyclic chain complexes. Torsion is an invariant defined for a based chain complexes. We denote by $(C_*, \mathbf{c}^* = \cup_i \mathbf{c}^i)$ the *based* chain complex:

$$C_* : 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

where each chain module C_i is a vector space over a field $\mathbb{F}$ and equipped with a basis $\mathbf{c}^i$. We are mainly interested in *acyclic* chain complexes which has trivial homology groups, i.e., $H_*(C_*) = \mathbf{0}$. The chain complex C_* also has a basis determined by the boundary operators ∂_i, which arises from the following decomposition of chain modules.

We suppose that a based chain complex $(C_*, \mathbf{c}^*)$ is acyclic. For each boundary operator ∂_i, we denote $\ker \partial_i \subset C_i$ by Z_i and the image of ∂_i by $B_i \subset C_{i-1}$. The chain module C_i is expressed as the direct sum of Z_i and the lift of B_i, denoted by $\tilde{B}_i$. Moreover we can rewrite the kernel Z_i as the image of boundary operator ∂_{i+1}:

$$\begin{aligned} C_i &= Z_i \oplus \tilde{B}_i \\ &= \partial_{i+1}\tilde{B}_{i+1} \oplus \tilde{B}_i \end{aligned}$$

where $Z_i = B_{i+1}$ is written as $\partial_{i+1}\tilde{B}_{i+1}$.

We denote by $\tilde{\mathbf{b}}^i$ a basis of $\tilde{B}_i$. Then the set $\partial_{i+1}(\tilde{\mathbf{b}}^{i+1}) \cup \tilde{\mathbf{b}}^i$ forms a new basis of the vector space C_i. We define the *torsion* of $(C_*, \mathbf{c}^*)$ as the following alternating product of

determinants of base change matrices:

$$\mathrm{Tor}(C_*, \mathbf{c}^*) = \prod_{i \geq 0} \left[\partial_{i+1}(\tilde{\mathbf{b}}^{i+1}) \cup \tilde{\mathbf{b}}^i \,/\, \mathbf{c}^i \right]^{(-1)^{i+1}} \in \mathbb{F}^* = \mathbb{F} \setminus \{0\} \tag{1}$$

where $[\partial_{i+1}(\tilde{\mathbf{b}}^{i+1}) \cup \tilde{\mathbf{b}}^i \,/\, \mathbf{c}^i]$ denotes the determinant of base change matrix from the given basis $\mathbf{c}^i$ to the new one $\partial_{i+1}(\tilde{\mathbf{b}}^{i+1}) \cup \tilde{\mathbf{b}}^i$.

Note that the right hand side is independent of the choice of bases $\tilde{\mathbf{b}}^i$. The alternating product in (1) is determinant by the based chain complex $(C_*, \mathbf{c}^*)$.

Reidemeister torsion for CW–complexes. We apply the torsion (1) of based chain complexes to the *twisted chain complexes* given by CW–complexes and homomorphisms from their fundamental groups to some linear groups. Let W denote a finite CW–complex and (V, ρ) a representation of $\pi_1(W)$, which means V is a vector space over $\mathbb{F}$ and ρ is a homomorphism from $\pi_1(W)$ into $\mathrm{GL}(V)$. We will call ρ a $\mathrm{GL}(V)$-representation of $\pi_1(W)$ simply.

Definition 2.1. We define the twisted chain complex $C_*(W; V_\rho)$ which consists of the twisted chain module as:

$$C_i(W; V_\rho) := V \otimes_{\mathbb{Z}[\pi_1(W)]} C_i(\widetilde{W}; \mathbb{Z})$$

where $\widetilde{W}$ is the universal cover of W and $C_i(\tilde{W}; \mathbb{Z})$ is a left $\mathbb{Z}[\pi_1(W)]$-module given by the covering transformation of $\pi_1(W)$. In taking the tensor product, we regard V as a right $\mathbb{Z}[\pi_1(W)]$-module under the homomorphism ρ^{-1}. We identify a chain $\boldsymbol{v} \otimes \gamma c$ with $\rho(\gamma)^{-1}(\boldsymbol{v}) \otimes c$ in $C_i(W; V_\rho)$.

We call $C_*(W; V_\rho)$ the twisted chain complex with the coefficient V_ρ. Choosing a basis of the vector space V, we give a basis of the twisted chain complex $C_*(W; V_\rho)$. To be more precise, let $\{e^i_1, \ldots, e^i_{m_i}\}$ be the set of i-dimensional cells of W and $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ a basis of V where $d = \dim_{\mathbb{F}} V$. Choosing a lift $\tilde{e}^i_j$ of each cell and taking tensor product with the basis of V, we have the following basis of $C_i(W; V_\rho)$:

$$\mathbf{c}^i(W; V) = \{\boldsymbol{v}_1 \otimes \tilde{e}^i_1, \ldots, \boldsymbol{v}_d \otimes \tilde{e}^i_1, \ldots, \boldsymbol{v}_1 \otimes \tilde{e}^i_{m_i}, \ldots, \boldsymbol{v}_d \otimes \tilde{e}^i_{m_i}\}.$$

We denote by $H_*(W; V_\rho)$ the homology group and call it *the twisted homology group* and say that ρ is acyclic if the twisted homology group vanishes. Regarding $C_*(W; V_\rho)$ as a based chain complex, we define the Reidemeister torsion for W and an acyclic representation (V, ρ) as the torsion of $C_*(W; V_\rho)$, i.e.,

$$\mathrm{Tor}(W; V_\rho) = \mathrm{Tor}(C_*(W; V_\rho), \mathbf{c}^*(W; V)) \in \mathbb{F}^* \tag{2}$$

up to a factor in $\{\pm \det(\rho(\gamma)) \,|\, \gamma \in \pi_1(W)\}$ since we have many choices of lifts $\tilde{e}^i_j$ and orders and orientations of cells e^i_j. We call $\mathrm{Tor}(W; V_\rho)$ the Reidemeister torsion of W and a $\mathrm{GL}(V)$-representation ρ.

Remark 2.2. We mention some well–definedness of the torsion (2):

- The acyclicity of $C_*(W; V_\rho)$ implies that the Euler characteristic of W is zero. Then the torsion (2) of $C_*(W; V_\rho)$ is independent of the choice of a basis in V.
- If we choose an $\mathrm{SL}(V)$-representation ρ with an even dimensional V, then the Reidemeister torsion $\mathrm{Tor}(W; V_\rho)$ has no indeterminacy.
- The Reidemeister torsion has an invariance under the conjugation of representations.

The following lemma for torsion (1) will be needed to deduce our surgery formula:

Lemma 2.3 (Multiplicativity Lemma). *Let* $0 \to (C'_*, \overline{\mathbf{c}}^*) \to (C_*, \mathbf{c}^*) \to (C''_*, \overline{\overline{\mathbf{c}}}^*) \to 0$ *be the short exact sequence of based chain complexes such that* $[\overline{\mathbf{c}}^i \cup \overline{\overline{\mathbf{c}}}^i / \mathbf{c}^i] = 1$ *for all i. Suppose that any two of the complexes are acyclic. Then the third one is also acyclic and the torsion of the three complexes are well-defined. Furthermore we have the next equality:*

$$\mathrm{Tor}(C_*, \mathbf{c}^*) = (-1)^{\sum_{i \geq 0} \beta'_{i-1} \beta''_i} \mathrm{Tor}(C'_*, \overline{\mathbf{c}}^*) \mathrm{Tor}(C''_*, \overline{\overline{\mathbf{c}}}^*)$$

where $\beta'_i = \dim_{\mathbb{F}} \partial C'_{i+1}$ *and* $\beta''_i = \dim_{\mathbb{F}} \partial C''_{i+1}$.

We refer to Milnor's survey [Mil66] and Turaev's book [Tur01] for more details on Reidemeister torsion.

2.2. **Higher dimensional Reidemeister torsion for** $\mathrm{SL}_2(\mathbb{C})$**-representations.** We will consider a sequence of the Reidemeister torsion of a finite CW-complex W which corresponds to the sequence of the $\mathrm{SL}_n(\mathbb{C})$-representations of $\pi_1(W)$, induced by an $\mathrm{SL}_2(\mathbb{C})$-representation. Let ρ be a homomorphism from $\pi_1(W)$ to $\mathrm{SL}_2(\mathbb{C})$. Then the pair $(\mathbb{C}^2, \rho)$ is a $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(W)$ by the standard action of $\mathrm{SL}_2(\mathbb{C})$ to $\mathbb{C}^2$. It is known that the symmetric product $\mathrm{Sym}^{n-1}(\mathbb{C}^2)$ and the induced action by $\mathrm{SL}_2(\mathbb{C})$ gives an n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. The symmetric product $\mathrm{Sym}^{n-1}(\mathbb{C}^2)$ can be identified with the vector space V_n of homogeneous polynomials on $\mathbb{C}^2$ with degree $n-1$, i.e.,

$$V_n = \mathrm{span}_{\mathbb{C}} \langle z_1^{n-1}, z_1^{n-2} z_2, \ldots, z_1^{n-k-1} z_2^k, \ldots, z_1 z_2^{n-2}, z_2^{n-1} \rangle$$

and the action of $A \in \mathrm{SL}_2(\mathbb{C})$ is expressed as

$$A \cdot p(z_1, z_2) = p(A^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) \quad \text{for} \quad p(z_1, z_2) \in V_n \tag{3}$$

We write (V_n, σ_n) for the representation given by the action (3) of $\mathrm{SL}_2(\mathbb{C})$ where σ_n denotes the homomorphism from $\mathrm{SL}_2(\mathbb{C})$ into $\mathrm{GL}(V_n)$. In fact the image of σ_n is contained in $\mathrm{SL}_n(\mathbb{C})$. We will see this after the definition of higher dimensional Reidemeister torsion.

We write ρ_n for the composition $\sigma_n \circ \rho$. Then we can take V_n as a coefficient of twisted chain complex for W since the vector space V_n is a right $\mathbb{Z}[\pi_1(W)]$-module of $\pi_1(W)$. We denote by $C_*(W; V_n)$ this twisted chain complex of W defined by (V_n, ρ_n). We will drop the subscript ρ_n in the coefficient for simplicity when no confusion can arise.

Definition 2.4. When the twisted chain complex $C_*(W; V_n)$ is acyclic, we define the higher dimensional Reidemeister torsion for W and ρ_n as $\mathrm{Tor}(W; V_n)$ and denote by $\mathrm{Tor}(W; \rho_n)$ since the coefficient V_n of $C_*(W; V_n)$ is determined by ρ and n.

Increasing n to infinity, we obtain the sequence of the Reidemeister torsion $\mathrm{Tor}(W; V_n)$ when $C_*(W; V_n)$ is acyclic. We will observe the asymptotic behaviors of these sequences for Seifert fibered spaces in the subsequent section.

We also review eigenvalues of the image $\sigma_n(A)$ for $A \in \mathrm{SL}_2(\mathbb{C})$. Let $a^{\pm 1}$ be the eigenvalues of A. By direct calculation, we can see that the eigenvalues of $\sigma_n(A) \in \mathrm{SL}_n(\mathbb{C})$ are given by $a^{-n+1}, a^{-n+3}, \ldots, a^{n-1}$, i.e., the weight space of σ_n is $\{-n+1, -n+3, \ldots, n-1\}$ and the multiplicity of each weight is 1.

Remark 2.5. If $n > 1$ is even (resp. odd), the eigenvalues of $\sigma_n(A)$ are the odd (resp. even) powers, from 1 (resp. 0) to $n-1$, of the eigenvalues of A. This implies that $\det \sigma_n(A) = 1$ for any $A \in \mathrm{SL}_2(\mathbb{C})$ and $n \geq 1$.

3. Surgery formula for the asymptotics of higher dimensional Reidemeister torsion

We will give a surgery formula to observe the asymptotic behavior of the sequence of higher dimensional Reidemeister torsions. Our surgery formula is based on Lemma 2.3 (the Multiplicativity Lemma) in Section 2.1. We will consider a connected compact orientable 3-manifold M with torus boundary $\partial M = T^2_1 \cup \ldots \cup T^2_m$. We denote a pair of meridian and longitude on T^2_j by (q_j, h_j), i.e.,$\pi_1(T^2_j) = \langle q_j, h_j \,|\, [q_j, h_j] = 1 \rangle$. By Dehn filling with slopes $\alpha_1/\beta_1, \ldots, \alpha_m/\beta_m$, we obtain a closed 3-manifold $M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$. Here a slope α_j/β_j is the unoriented isotopy class of the essential simple loop $\alpha_j q_j + \beta_j h_j$ on the j-th boundary component T^2_j, i.e.,,

$$M(\tfrac{\alpha_1}{\beta_1}, \ldots, \tfrac{\alpha_m}{\beta_m}) = M \cup (\cup_{j=1}^m D^2_j \times S^1_j) \quad \text{where} \quad \partial D^2_j \times \{*\} \sim \alpha_j q_j + \beta_j h_j \quad (\forall j).$$

Our purpose is to express the higher dimensional Reidemeister torsion of resulting manifolds $M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$ by those of M and solid tori $S_j = D^2_j \times S^1_j$ $(j = 1, \ldots, m)$. We start with a homomorphism ρ from $\pi_1(M)$ to $\mathrm{SL}_2(\mathbb{C})$. When ρ satisfies that the equations $\rho(q_j)^{\alpha_j}\rho(h_j)^{\beta_j} = I$, it extends to a homomorphism of the fundamental group of the resulting manifold and also defines higher dimensional representations ρ_n of $\pi_1(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}))$. Then we can consider the Mayer–Vietoris exact sequence with the coefficient V_n:

$$0 \to \oplus_{j=1}^m C_*(T^2_j; V_n) \to C_*(M; V_n) \oplus (\oplus_{j=1}^m C_*(S_j; V_n)) \to C_*(M(\tfrac{\alpha_1}{\beta_1}, \ldots, \tfrac{\alpha_m}{\beta_m}); V_n) \to 0. \tag{4}$$

If the left and middle parts in the Mayer–Vietoris sequence (4) are acyclic, then the higher dimensional Reidemeister torsion of $M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$ is expressed as

$$\mathrm{Tor}(M(\tfrac{\alpha_1}{\beta_1}, \ldots, \tfrac{\alpha_m}{\beta_m}); \rho_n) = \pm \mathrm{Tor}(M; \rho_n) \cdot \prod_{j=1}^m \mathrm{Tor}(S_j; \rho_n) \cdot \prod_{j=1}^m \mathrm{Tor}(T^2_j; \rho_n)^{-1}. \tag{5}$$

by Lemma 2.3 (Multiplicativity Lemma).

We have seen that if n is odd, then the image $\sigma_n(A)$ always has the eigenvalue 1 for any $A \in \mathrm{SL}_2(\mathbb{C})$. This implies that the twisted chain complex $C_*(S_j; V_n)$ is never acyclic when $n = 2N-1$ (this will be seen in the following Subsection 3.1). Hence we will focus on even dimensional representations ρ_{2N} to apply the Multiplicativity Lemma of acyclic Reidemeister torsion..

In Subsection 3.1, we will give equivalent conditions for the twisted chain complexes for T^2_j and S_j to be acyclic for all $2N$. At least, We have to work on our surgery formula under the resulting conditions. We will deduce from Eq. (5) a surgery formula for the asymptotic behaviors of the sequences induced by the higher dimensional Reidemeister torsion of $M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$ and M in Subsection 3.2. The last Subsection 3.3 gives the example of our surgery formula in the case that M is a torus knot exterior.

3.1. Acyclicity conditions for the boundary and solid tori.

First, we review the twisted chain complexes of T^2 and $D^2 \times S^1$. The torsion of $D^2 \times S^1$ coincides with that of core $\{0\} \times S^1$ since they are simple homotopy equivalent. We consider the twisted chain complexes of S^1 instead of $D^2 \times S^1$. Under the cell decomposition

$$S^1 = e^0 \cup e^1, \quad \text{and} \quad T^2 = e^0 \cup e^1_1 \cup e^1_2 \cup e^2,$$

the twisted chain complexes with the coefficient V_n of T^2 and ℓ are described as follows:

$$C_*(S^1; V_n) \colon\ 0 \to C_1(S^1; V_n) = V_n \xrightarrow{L-I} C_0(S^1; V_n) = V_n \to 0,$$

$$C_*(T^2; V_n) \colon\ 0 \to C_2(T^2; V_n) = V_n \xrightarrow{\partial_2} C_1(T^2; V_n) = V_n^{\oplus 2} \xrightarrow{\partial_1} C_0(T^2; V_n) = V_n \to 0,$$

$$\partial_2 = \begin{pmatrix} -H+I \\ Q-I \end{pmatrix}, \quad \partial_1 = (Q-I, H-I)$$

where L, Q and H denote $\mathrm{SL}_n(\mathbb{C})$-matrices corresponding the simple closed loops $\ell = e^0 \cup e^1$ in S^1, $q = e^0 \cup e^1_1$ and $h = e^0 \cup e^1_2$ in T^2.

The twisted homology group $H_1(S^1; V_n)$ is the eigenspace of L for the eigenvalue 1. As mentioned in Remark 2.5, if n is odd, then the $\mathrm{SL}_n(\mathbb{C})$-matrix L always has the eigenvalue 1. Hence $C_*(S^1; V_{2N-1})$ can not be acyclic. Here and subsequently, we focus only on the twisted chain complexes given by even dimensional vector spaces V_{2N}.

It is known that every abelian subgroup in $\mathrm{SL}_2(\mathbb{C})$ is moved by conjugation into either the maximal abelian subgroups Hyp or Para:

$$\mathrm{Hyp} := \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \,\middle|\, z \in \mathbb{C} \setminus \{0\} \right\},\ \mathrm{Para} := \left\{ \begin{pmatrix} \pm 1 & w \\ 0 & \pm 1 \end{pmatrix} \,\middle|\, w \in \mathbb{C} \right\}.$$

Since the conjugation of representations induces an isomorphism of twisted homology groups, we can assume that the images of $\pi_1(S^1)$ and T^2 by $\mathrm{SL}_2(\mathbb{C})$-representations are contained in Hyp or Para.

We describe the acyclicity conditions for the twisted chain complexes $C_*(S^1; V_{2N})$ and $C_*(T^2; V_{2N})$ by the terminologies of $\mathrm{SL}_2(\mathbb{C})$ for the matrices corresponding to generators of $\pi_1(S^1)$ and $\pi_1(T^2)$.

Proposition 3.1. *Let ρ be an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(S^1) = \langle \ell \rangle$. The composition $\rho_{2N} = \sigma_{2N} \circ \rho$ is acyclic for all $N \geq 1$ if and only if $\rho(\ell)$ is neither of odd order nor parabolic with trace* 2*.*

Proof. The dimensions of $H_0(S^1; V_{2N})$ and $H_1(S^1; V_{2N})$ are same since the Euler characteristic of S^1 is zero. The homology group $H_1(S^1; V_{2N})$ is the eigenspace of $L = \rho_{2N}(\ell)$ for the eigenvalue 1. The acyclicity of ρ_{2N} is equivalent for the $\mathrm{SL}_{2N}(\mathbb{C})$-matrix L not to have the eigenvalue 1.

If $\rho(\gamma) \in \mathrm{Hyp}$ is not of odd order, then the eigenvalues of L forms $\{e_\ell^{\pm(2k-1)} \mid k = 1, \ldots, N\}$ where $e_\ell^{\pm 1}$ are the eigenvalues of $\rho(\ell)$. Since e_ℓ is not of odd order, the $\mathrm{SL}_{2N}(\mathbb{C})$-element L does not have the eigenvalue 1 for all N. Thus ρ_{2N} is acyclic for all N. If $\rho(\ell) \in \mathrm{Para}$ has trace -2, then the eigenvalue of L is just -1 for all N. The $\mathrm{SL}_{2N}(\mathbb{C})$-representation ρ_{2N} is also acyclic for all N.

Conversely suppose that $\rho(\ell)$ has the order $2k_\ell - 1$. Then the set of eigenvalues of L contain 1 when $N \geq k_\ell$. The twisted homology group $H_1(S^1; V_{2N})$ is not trivial for $N \geq k_\ell$. Suppose that $\rho(\ell) \in \mathrm{Para}$ has trace 2. Then the eigenvalue of L is just 1 for all N. The twisted homology group $H_1(S^1; V_{2N})$ is not trivial for all N. □

Proposition 3.2. *Let ρ be an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(T^2) = \langle q, h \mid [q, h] = 1 \rangle$. The composition $\rho_{2N} = \sigma_{2N} \circ \rho$ is acyclic if and only if either $\rho(q)$ or $\rho(h)$ is neither of odd order nor parabolic with the trace* 2*.*

Proof. The homology group $H_1(T^2; V_{2N})$ is the common eigenspace of $Q = \rho_{2N}(q)$ and $H = \rho_{2N}(h)$ for the eigenvalue 1. Since the Euler characteristic of T^2 is zero, by Poincaré duality, the twisted homology group $H_*(T^2; V_{2N})$ vanishes if and only if $H_2(T^2; V_{2N}) = 0$.

The acyclicity of ρ_{2N} is equivalent to exist no common eigenspace of Q and H for the eigenvalue 1.

If $\rho(q)$ is neither of odd order nor parabolic with trace 2, then Q does not have the eigenvalue 1 for all N as in the proof of Proposition 3.1. We have no common eigenspace of Q and H for the eigenvalue 1. Hence ρ_{2N} is acyclic for all N.

Conversely suppose that $\rho(q)$ and $\rho(h)$ have the orders $2k_q - 1$ and $2k_h - 1$. Then at the weight $(2k_q - 1)(2k_h - 1)$, we have a common eigenspace of Q and H for the eigenvalue 1 when N is sufficiently large. Thus ρ_{2N} is not acyclic for sufficiently large N. Suppose that $\rho(q)$ and $\rho(h)$ are parabolic with trace 2. Then Q and H are always upper triangular matrix whose all diagonal entries are 1. We have a common eigenspace of Q and H for the eigenvalue 1. Hence ρ_{2N} is not acyclic for all N. □

Remark 3.3. One can show that the $\mathrm{SL}_{2N-1}(\mathbb{C})$-representation ρ_{2N-1} of $\pi_1(T^2)$ is not acyclic for all $N \geq 1$ by the similar argument in Proposition 3.2.

3.2. **Surgery formula for the asymptotic behaviors.** We show a surgery formula for the asymptotic behaviors of higher dimensional Reidemeister torsions of a closed 3-manifold $M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}) = M \cup (\cup_{j=1}^m) D^2_j \times S^1_j$. To apply Lemma 2.3 (Multiplicativity Lemma) for acyclic chain complexes, we assume that the following acyclicity conditions for the twisted chain complexes $\partial M = \cup_{j=1}^m T^2_j$ and solid tori $S_j = D^2_j \times S^2_j$ with the presentations of fundamental groups:

$$\pi_1(T^2_j) = \langle q_j, h_j \,|\, [q_j, h_j] = 1 \rangle, \quad \pi_1(S_j) = \langle \ell_j \rangle.$$

Definition 3.4 (Acyclicity conditions)**.** Let ρ be a homomorphism from $\pi_1(M)$ to $\mathrm{SL}_2(\mathbb{C})$ such that $\rho(q_j^{\alpha_j} h_j^{\beta_j}) = I$ for all $j = 1, \ldots m$. We use the same symbol ρ for the induced homomorphism of $\pi_1(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}))$ and assume that

(i) Either $\rho(q_j)$ or $\rho(h_j)$ is of even order for all j and;
(ii) The order of $\rho(\ell_j)$ is also even for all j.

We will call the above conditions (i) & (ii) *the acyclicity conditions*.

Remark 3.5. The acyclicity condition guarantees that all twisted chain complexes of T^2_j and S_j are acyclic. Our acyclicity conditions are more restricted as compared with the conditions in Propositions 3.2 & 3.1. However in the case that the resulting manifold $M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$ is a Seifert fibered space, it is reasonable to assume our conditions as seen in Section 4.

Under the acyclic conditions, if a $\mathrm{SL}_{2N}(\mathbb{C})$-representation ρ_{2N} of $\pi_1(M)$ is acyclic, then we can express the higher dimensional Reidemeister torsion of $M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$ as

$$\mathrm{Tor}(M(\tfrac{\alpha_1}{\beta_1}, \ldots, \tfrac{\alpha_m}{\beta_m}); \rho_{2N}) = \mathrm{Tor}(M; \rho_{2N}) \cdot \prod_{j=1}^m \mathrm{Tor}(S_j; \rho_{2N}) \cdot \prod_{j=1}^m \mathrm{Tor}(T^2_j; \rho_{2N})^{-1}. \tag{6}$$

by applying Lemma 2.3 (Multiplicativity Lemma). Note that every integer β'_i in Lemma 2.3 is even from the acyclicity of $C_*(T^2_j; V_{2N})$.

Then the asymptotics of $\log|\mathrm{Tor}(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|$ for $2N$ is determined by that of $\log|\mathrm{Tor}(M; \rho_{2N})|$ as follows.

Theorem 3.6. *Let ρ be an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M)$ satisfying $\rho(q_j^{\alpha_j} h_j^{\beta_j}) = I$ and the acyclicity conditions in Definition 3.4. Suppose that ρ_{2N} of $\pi_1(M)$ is acyclic for all N. Then the asymptotics of $\log|\mathrm{Tor}(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|$ is expressed as follows:*

(i) $\displaystyle\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{\alpha_1}{\beta_1},\ldots,\frac{\alpha_m}{\beta_m});\rho_{2N})|}{(2N)^2} = \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{(2N)^2}$,

(ii) $\displaystyle\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{\alpha_1}{\beta_1},\ldots,\frac{\alpha_m}{\beta_m});\rho_{2N})|}{2N} = \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} - \log 2 \sum_{j=1}^{m} \frac{1}{\lambda_j}$

where $2\lambda_j$ is the order of $\rho(\ell_j)$ and ℓ_j is the homotopy class of $\{0\}\times S^1_j \subset S_j$.

Proof. By Eq. (6), the logarithm $\log|\mathrm{Tor}(M(\frac{\alpha_1}{\beta_1},\ldots,\frac{\alpha_m}{\beta_m});\rho_{2N})|$ is expressed as

$$\begin{aligned}&\log|\mathrm{Tor}(M(\tfrac{\alpha_1}{\beta_1},\ldots,\tfrac{\alpha_m}{\beta_m});\rho_{2N})|\\ &\quad= \log|\mathrm{Tor}(M;\rho_{2N})| + \sum_{J=1}^{m}\log|\mathrm{Tor}(S_j;\rho_{2N})| - \sum_{j=1}^{m}\log|\mathrm{Tor}(T^2_j;\rho_{2N})|.\end{aligned}$$

Applying the following Propositions 3.7 & 3.8, we obtain Theorem 3.6. □

Proposition 3.7. *Let ρ be an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(T^2) = \langle q,h\,|\,[q,h]=1\rangle$ such that either $\rho(q)$ or $\rho(h)$ is of even order. Then $\mathrm{Tor}(T^2;\rho_{2N}) = 1$ for all $N \geq 1$.*

Proof. This follows form the direct calculation. □

Proposition 3.8. *Let ρ be an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(S^1) = \langle \ell = 1\rangle$ such that $\rho(\ell)$ is of even order. Then we have the following limits of $\log|\mathrm{Tor}(S^1;\rho_{2N})|$:*

(i) $\displaystyle\lim_{N\to\infty}\frac{\log|\mathrm{Tor}(S^1;\rho_{2N})|}{(2N)^2} = 0$,

(ii) $\displaystyle\lim_{N\to\infty}\frac{\log|\mathrm{Tor}(S^1;\rho_{2N})|}{2N} = \frac{-\log 2}{\lambda}$

where 2λ is the order of $\rho(\ell)$.

Proof. We begin by computing the higher dimensional Reidemeister torsion $\mathrm{Tor}(S^1;\rho_{2N})$. We can express the Reidemeister torsion as $\mathrm{Tor}(S^1;\rho_{2N}) = \det(\rho_{2N}(\ell)-I)^{-1}$. The set of the eigenvalues of $\rho_{2N}(\ell)$ is obtained from the eigenvalues $e^{\pm\pi\eta\sqrt{-1}/\lambda}$ of $\rho(\ell)$, where η is odd and η and λ are coprime. It turns into $\{e^{\pm\pi(2k-1)\eta\sqrt{-1}/\lambda}\,|\,k=1,\ldots,N\}$. Hence the Reidemeister torsion $\mathrm{Tor}(S^1;\rho_{2N})$ turns out

$$\begin{aligned}\mathrm{Tor}(S^1;\rho_{2N}) &= \prod_{k=1}^{m}\{(e^{\pi(2k-1)\eta\sqrt{-1}/\lambda}-1)((e^{-\pi(2k-1)\eta\sqrt{-1}/\lambda}-1))\}^{-1}\\ &= \prod_{k=1}^{N}\left(2\sin\frac{\pi(2k-1)\eta}{2\lambda}\right)^{-2}\end{aligned}$$

The logarithm $\log|\mathrm{Tor}(S^1;\rho_{2N})|$ is expressed as

$$\log|\mathrm{Tor}(S^1;\rho_{2N})| = 2N\log 2^{-1} + 2\sum_{k=1}^{N}\log\left|\sin\frac{\pi(2k-1)\eta}{2\lambda}\right|^{-1}. \tag{7}$$

We can now proceed to compute the limits (i) & (ii).

(i) From the following inequality

$$\left|\sin\frac{\pi}{2\lambda}\right| \leq \left|\sin\frac{\pi(2k-1)\eta}{2\lambda}\right| \leq 1$$

it follows that

$$N\log\left|\sin\frac{\pi}{2\lambda}\right|^{-1} \geq \sum_{k=1}^{N}\log\left|\sin\frac{\pi(2k-1)\eta}{2\lambda}\right|^{-1} \geq 0. \tag{8}$$

By the inequality (8) and explicit form (7) of $\log|\operatorname{Tor}(S^1;\rho_{2N})|$, we can assert

$$\lim_{N\to\infty}\frac{\log|\operatorname{Tor}(S^1;\rho_{2N})|}{(2N)^2}=0.$$

(ii) We can express the limit (ii) as

$$\lim_{N\to\infty}\frac{\log|\operatorname{Tor}(S^1;\rho_{2N})|}{2N}=\log 2^{-1}+\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\log\left|\sin\frac{\pi(2k-1)\eta}{2\lambda}\right|^{-1}. \tag{9}$$

The second term in the right hand side of (9) can be rewritten as

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\log\left|\sin\frac{\pi(2k-1)\eta}{2\lambda}\right|^{-1}=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\log\left|\sin\left(\frac{\pi\eta}{2\lambda}+\frac{\pi(k-1)\eta}{\lambda}\right)\right|^{-1}$$

The sequence $\{\log|\sin(\pi\eta/(2\lambda)+\pi(k-1)\eta/\lambda)|^{-1}\}_N$ has the minimum period λ since η and λ are coprime. By Lemma 3.10, we can rewrite as

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\log\left|\sin\left(\frac{\pi\eta}{2\lambda}+\frac{\pi(k-1)\eta}{\lambda}\right)\right|^{-1}=\frac{1}{\lambda}\sum_{k=1}^{\lambda}\log\left|\sin\left(\frac{\pi\eta}{2\lambda}+\frac{\pi(k-1)\eta}{\lambda}\right)\right|^{-1}.$$

The right hand side of (9) turns into

$$\begin{aligned}\log 2^{-1}+\frac{1}{\lambda}\sum_{k=1}^{\lambda}\log\left|\sin\left(\frac{\pi\eta}{2\lambda}+\frac{\pi(k-1)\eta}{\lambda}\right)\right|^{-1}&=\frac{1}{\lambda}\log\prod_{k=1}^{\lambda}\left|2\sin\left(\frac{\pi\eta}{2\lambda}+\frac{\pi(k-1)\eta}{\lambda}\right)\right|^{-1}\\&=\frac{1}{\lambda}\log\left|2\sin\left(\frac{\pi\eta}{2}\right)\right|^{-1}\end{aligned}$$

by $|2\sin(n\theta)|=\prod_{k=0}^{n-1}|2\sin(\theta+k\pi/n)|$. Therefore we obtain the limit

$$\lim_{N\to\infty}\frac{\log|\operatorname{Tor}(S^1;\rho_{2N})|}{2N}=\frac{-\log 2}{\lambda}$$

since η is odd. □

Remark 3.9. We obtain $|2\sin(n\theta)|=\prod_{k=0}^{n-1}|2\sin(\theta+k\pi/n)|$ from substituting $z=e^{-2\theta\sqrt{-1}}$ to $|z^n-1|=\prod_{k=0}^{n-1}|z-e^{2\pi k\sqrt{-1}/n}|$.

Lemma 3.10. *Let $\{a_k\,|\,a_k\in\mathbb{R}\}_{k\geq 1}$ be a sequence such that $a_k\geq 0$ and $a_{k+N_0}=a_k$. Then we have the following limit:*

$$\lim_{N\to\infty}\frac{a_1+\cdots+a_N}{N}=\frac{a_1+\cdots+a_{N_0}}{N_0}.$$

Proof. It follows that $\left[\frac{N}{N_0}\right]\sum_{k=1}^{N_0}a_k\leq\sum_{k=1}^{N}a_k\leq\left(\left[\frac{N}{N_0}\right]+1\right)\sum_{k=1}^{N_0}a_k$ where $[\,\cdot\,]$ denotes the Gaussian symbol. □

Remark 3.11. In Lemma 3.10, it is not required that the period N_0 is minimum. However we have the same average for any period N_0.

3.3. **Example for Dehn fillings of torus knot exteriors.** We give examples of Theorem 3.6 for integral surgeries along torus knots in S^3. Let M be the (p,q)-torus knot exterior which is obtained by removing an open tubular neighbourhood of the knot from S^3. After gluing a solid torus along the slope $1/n$ ($n \in \mathbb{Z}$) on ∂N, we have an integral homology 3-sphere $M(\frac{1}{n})$. Since we consider the (p,q)-torus knot exterior, the resulting manifold $M(\frac{1}{n})$ is a Brieskorn homology 3-sphere of index $(p, q, pqn \pm 1)$. Here the sign in $pqn \pm 1$ depends on the orientation of the preferred longitude on ∂M.

The (p,q)-torus knot group admits the following presentation:

$$\pi_1(M) = \langle x, y \,|\, x^p = y^q \rangle$$

In this presentation, we can express a pair of meridian m and longitude ℓ as

$$m = x^{-r} y^s, \quad \ell = m^{pq} x^{-p}$$

where r and s are integers satisfying that $ps - qr = 1$. Then the Brieskorn homology sphere $M(\frac{1}{n})$ has the index $(p, q, pqn+1)$. We consider irreducible $\mathrm{SL}_2(\mathbb{C})$-representations ρ of $\pi_1(M)$ such that $\rho(m\ell^n) = I$, i.e., they extend to irreducible $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(M(\frac{1}{n})) = \langle x, y \,|\, x^p = y^q, m\ell^n = 1 \rangle$. Here irreducible means that there are no common eigenspaces among all elements in $\rho(\pi_1(M))$. Under the assumption of irreducibility for ρ, the central element $x^p (= y^q)$ must be sent to $\pm I$. The requirement that $\rho(m\ell^n) = I$ turns into $\rho(m)^{npq+1} = \pm I$. Hence we have the constrains on the order of $\rho(x)$, $\rho(y)$ and $\rho(m)$ for every irreducible $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M(\frac{1}{n}))$.

The conjugacy classes of irreducible $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(M(\frac{1}{n}))$ form a finite set. Each member of the finite set corresponds to triples of integers. This was shown by D. Johnson [Joh] and he also gave the explicit form of Reidemeister torsion for acyclic $\mathrm{SL}_2(\mathbb{C})$-representations as follows.

Theorem 3.12 (Johnson)**.** *The conjugacy classes of irreducible* $\mathrm{SL}_2(\mathbb{C})$*-representations* ρ *of* $\pi_1(M(\frac{1}{n}))$ *are given by triples* (a, b, c) *such that*

(i) $0 < a < p$, $0 < b < q$, $a \equiv b \mod 2$,
(ii) $0 < c < N = |pqn + 1|$, $c \equiv na \mod 2$,
(iii) $\mathrm{tr}\,\rho(x) = 2\cos \pi a/p$,
(iv) $\mathrm{tr}\,\rho(y) = 2\cos \pi b/q$,
(v) $\mathrm{tr}\,\rho(m) = 2\cos \pi c/N$.

The Reidemeister torsion is given by

$$\mathrm{Tor}(M(\tfrac{1}{n}); \rho) = \begin{cases} 2^{-4} \sin^{-2} \frac{\pi a}{2p} \sin^{-2} \frac{\pi b}{2q} \sin^{-2} \frac{\pi(cpq - N)}{2N} & a \equiv b \equiv 1, c \equiv n \mod 2 \\ 0 \quad \textit{(non-acyclic)} & a \equiv b \equiv 0 \textit{ or } c \not\equiv n \mod 2 \end{cases}$$

for $\rho \in (a, b, c)$.

In the remainder of this subsection, we denote by (a, b, c) the corresponding conjugacy class of irreducible $\mathrm{SL}_2(\mathbb{C})$-representations. We also refer to [Fre92] for the Reidemeister torsion of Brieskorn homology 3-spheres.

Remark 3.13. The parameters a and b determine the image of the central element $x^p (= y^q)$ by $(-I)^a = ((-I)^b)$.

To apply Theorem 3.6, we need to find

- a condition on (a, b, c) for all $\rho_{2N}|_{\pi_1(M)}$ to be acyclic and
- the orders of $\mathrm{SL}_2(\mathbb{C})$-elements in the acyclicity conditions (Definition 3.4).

The author has shown in [Yam, Proposition 3.1] that $\rho|_{\pi_1(M)}$ induces an acyclic $\mathrm{SL}_{2N}(\mathbb{C})$-representation for all N if and only if the parameters a and b of $\rho|_{\pi_1(M)}$ satisfy that $a \equiv b \equiv 1 \bmod 2$.

Since the surgery slope for $M(\frac{1}{n}) = M \cup D^2 \times S^1$ is $1/n$, the homotopy class of the core $\{0\} \times S^1$ is given by $\ell^{\pm 1}$ in $\pi_1(M(\frac{1}{n}))$. To check the acyclicity conditions, we only need to find the order of $\rho(\ell)$. Let ρ be in the conjugacy class (a, b, c) such that $a \equiv b \equiv 1 \bmod 2$. Then it follows from $c \equiv na \bmod 2$ that

$$\rho(\ell)^N = (-I)^{pqc-N} = -I.$$

Hence the eigenvalues of $\rho(\ell)$ are given by $e^{\pm\pi\eta\sqrt{-1}/N}$ for some odd integer η, which shows that $\rho(\ell)$ is of even order.

Let us apply Theorem 3.6 to the Brieskorn homology 3-sphere $M(\frac{1}{n})$. We obtain

$$\lim_{N\to\infty} \frac{\mathrm{Tor}(M(\frac{1}{n});\rho_{2N})}{(2N)^2} = \lim_{N\to\infty} \frac{\mathrm{Tor}(M;\rho_{2N})}{(2N)^2} \tag{10}$$

$$\lim_{N\to\infty} \frac{\mathrm{Tor}(M(\frac{1}{n});\rho_{2N})}{2N} = \lim_{N\to\infty} \frac{\mathrm{Tor}(M;\rho_{2N})}{2N} - \frac{\log 2}{N'} \tag{11}$$

where $2N'$ is the order of $\rho(\ell)$. Note that it is seen from the g.c.d. $(p, N) = (q, N) = 1$ that $N' = N/(c, N)$.

It has shown in [Yam, Theorem 4.2] that the right hand side in (10) vanishes. The higher Reidemeister torsion of the torus knot exterior N is expressed as, by [Yam, Proposition 4.1],

$$\mathrm{Tor}(M;\rho_{2N}) = \frac{2^{2N}}{\prod_{k=1}^{N} 4^2 \sin^2 \frac{\pi(2k-1)a}{2p} \sin^2 \frac{\pi(2k-1)b}{2q}}.$$

By a similar argument to the proof of Proposition 3.8, we can see that the limit in the right hand of (11) turns out

$$\lim_{N\to\infty} \frac{\mathrm{Tor}(M;\rho_{2N})}{2N} = \left(1 - \frac{1}{p'} - \frac{1}{q'}\right) \log 2$$

where $p' = p/(p, a)$ and $q' = q/(q, b)$.

Theorem 3.14. *The growth of* $\log|\mathrm{Tor}(M(\frac{1}{n});\rho_{2N})|$ *has the same order as* $2N$ *for any irreducible representation* ρ *of* $\pi_1(M(\frac{1}{n}))$*. Moreover if* ρ *is contained in the conjugacy class* (a, b, c)*, then the leading coefficient in* $2N$ *converges as*

$$\lim_{N\to\infty} \frac{\mathrm{Tor}(M(\frac{1}{n});\rho_{2N})}{2N} = \left(1 - \frac{1}{p'} - \frac{1}{q'} - \frac{1}{N'}\right) \log 2$$

where $p' = p/(a, p)$*,* $q' = q/(b, q)$ *and* $N' = N/(c, N)$*.*

In the case that $(a, p) = (b, q) = (c, N) = 1$*, the leading coefficient converges to the maximum* $(1 - 1/p - 1/q - 1/N) \log 2$

4. Asymptotics of higher dimensional Reidemeister torsion for Seifert fibered spaces

We will apply Theorem 3.6 to Seifert fibered spaces and study the asymptotic behaviors of their higher dimensional Reidemeister torsions. We will see the growth of the logarithm of higher dimensional Reidemeister torsion has the same order as the dimension of representations and we will also give the limit of the leading coefficient.

The limits of the leading coefficients are determined by each component in the $\mathrm{SL}_2(\mathbb{C})$-representation space of the fundamental group of a Seifert fibered space, that is to say, we obtain a locally constant function on the $\mathrm{SL}_2(\mathbb{C})$-representation space. From the invariance

of Reidemeister torsion under the conjugation of representations, we also obtain a locally constant function on the character variety of the fundamental group of a Seifert fibered space.

We will focus on SU(2)-character varieties for Seifert fibered homology spheres and describe explicit values of the locally constant functions. Our calculation shows that these locally constant functions take the maximal values on the top dimensional components and they are given by $-\chi \log 2$ where χ is the Euler characteristic of the base orbifold of a Seifert fibered homology sphere.

We start with review on Seifert fibered space in 4.1. Subsection 4.2 shows the application of Theorem 3.6 to Seifert fibered spaces. We will observe the relation between limits of the leading coefficients and components in SU(2)-character varieties in Subsection 4.3.

4.1. **Seifert fibered spaces.** A Seifert fibered space is referred as an S^1-fibration over a closed 2-orbifold. We consider the orientable Seifert fibered space given by the following Seifert index:

$$\{b, (o, g); (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}.$$

where $\alpha_j \geq 2$ ($j = 1, \ldots, m$) and each pair α_j and β_j is coprime. For a Seifert index, We refer to [NJ81, Orl72].

We can view a Seifert fibered space as an S^1-bundle over a closed orientable surface Σ with $m+1$ exceptional fibers, where the genus of Σ is g. From this viewpoint, we can decompose a Seifert fibered space into tubular neighbourhoods of exceptional fibers and their complement. Set $\Sigma_* = \Sigma \setminus (D^2_0 \cup \ldots \cup D^2_m)$ where $D^2_0, \ldots, D^2_m$ are disjoint open disks in Σ. Let M be the trivial S^1-bundle $\Sigma_* \times S^1$. We have a canonical decomposition of the Seifert fibered space as

$$\begin{aligned} &M \cup (S_0 \cup S_1 \cup \cdots \cup S_m) \\ &= M(\tfrac{1}{-b}, \tfrac{\alpha_1}{\beta_1}, \ldots, \tfrac{\alpha_m}{\beta_m}). \end{aligned}$$

The solid torus S_0 corresponds to the triviality obstruction b and the others S_j ($1 \leq j \leq m$) correspond to the exceptional fibers with the index (α_j, β_j). Then the fundamental group of $M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$ admits the following presentation:

$$\begin{aligned} (12)\quad &\pi_1(M(\tfrac{1}{-b}, \tfrac{\alpha_1}{\beta_1}, \ldots, \tfrac{\alpha_m}{\beta_m})) \\ &= \langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots . q_m, h \,|\, [a_i, h] = [b_i, h] = [q_j, h] = 1, \\ &\qquad q_j^{\alpha_j} h^{\beta_j} = 1, q_1 \cdots q_m [a_1, b_1] \cdots [a_g, b_g] = h^b \rangle \end{aligned}$$

where a_i and b_j correspond to generators of $\pi_1(\Sigma)$ and q_j is the corresponding to the circle $\partial D^2_j \subset \Sigma$ and h is the homotopy class of a regular fiber in M. Note that the presentation of $\pi_1(M)$ is given by $\langle a_1, b_1, \ldots, a_g, b_g, q_0, q_1, \ldots . q_m, h \,|\, [a_i, h] = [b_i, h] = [q_j, h] = 1, q_1 \cdots q_m [a_1, b_1] \cdots [a_g, b_g] = q_0 \rangle$.

We review the acyclicity of $\mathrm{SL}_n(\mathbb{C})$-representations of the fundamental group of M. We are supposed to consider the sequences of Reidemeister torsions for acyclic chain complexes $C_*(M; V_{2N})$ ($N \geq 1$) derived form an $\mathrm{SL}_2(\mathbb{C})$-representation ρ of $\pi_1(M)$. It was shown in T. Kitano [Kit94] that $C_*(M; \mathbb{C}^n)$ is acyclic if and only if $\rho_n(h) = -I$ where $(\mathbb{C}^n, \rho_n)$ is an $\mathrm{SL}_n(\mathbb{C})$-representation of $\pi_1(M)$.

We also touch the first homology groups of Seifert fibered spaces since we will consider Seifert fibered homology spheres in Subsection 4.3. It is known that the first homology group of $M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$ is expressed as $H_1(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \mathbb{Z}) \simeq \mathbb{Z}^{2g} \oplus T$ where T is a finite abelian group with the order $\alpha_1 \cdots \alpha_m |b + \sum_{j=1}^m \beta_j/\alpha_j|$ if $b + \sum_{j=1}^m \beta_j/\alpha_j$ is not zero.

In the case that $b + \sum_{j=1}^m \beta_j/\alpha_j = 0$, the homology group $H_1(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_m}{\beta_m}); \mathbb{Z})$ has the free rank $2g+1$. Hence, for any Seifert fibered homology sphere $M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_m}{\beta_m})$, the genus of the base orbifold is zero and we have the equation:

$$\alpha_1 \cdots \alpha_m \Big(b + \sum_{j=1}^m \frac{\beta_j}{\alpha_j} \Big) = 1,$$

in particular, which implies that α_j are pairwise coprime.

4.2. **The asymptotic behavior of higher dimensional Reidemeister torsions for Seifert fibered spaces.** We apply Theorem 3.6 to a Seifert fibered space $M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_m}{\beta_m})$ for $\mathrm{SL}_2(\mathbb{C})$-representations ρ which satisfy $\rho(h) = -I$, that is to say, the central element h is sent to the non-trivial central element $-I$ of $\mathrm{SL}_2(\mathbb{C})$. We begin with observing the acyclicity for M, ∂M and the solid tori S_j and confirm that the condition that $\rho(h) = -I$ is only needed in our situation.

It is shown from [Kit96, the proof of Proposition 3.1] that $C_*(M; V_{2N})$ is acyclic if and only if $\rho_{2N}(h) = -I$. It holds for all N that $\rho_{2N}(h) = -I$ under the assumption that $\rho(h) = -I$ since every weight of σ_{2N} for the eigenvalues of $\rho(h)$ is odd. Also we have shown that such an $\mathrm{SL}_2(\mathbb{C})$-representation ρ satisfies the acyclicity condition (i) for all boundary components T_j^2 since $\pi_1(T_j^2)$ is presented as

$$\pi_1(T_j^2) = \langle q_j, h \,|\, [q_j, h] = 1 \rangle$$

and $\rho(h)$ has the order 2. Furthermore we can see that all conditions in our surgery formula (Theorem 3.6) are satisfied under the assumption that $\rho(h) = -I$.

Proposition 4.1. *Suppose that an* $\mathrm{SL}_2(\mathbb{C})$*-representation* ρ *of* $\pi_1(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_m}{\beta_m}))$ *sends* h *to* $-I$*. Then* ρ *satisfies the acyclicity conditions in Definition 3.4 and the restriction of* ρ *gives an acyclic twisted chain complex* $C_*(M; V_{2N})$ *for all* N*.*

Proof. It remains to prove that the acyclicity for the twisted chain complexes of S_j (the condition (ii) in Definition 3.4). This follows from the following Lemma 4.2. □

Lemma 4.2. *Let* ρ *be an* $\mathrm{SL}_2(\mathbb{C})$*-representations of* $\pi_1(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_m}{\beta_m}))$ *such that* $\rho(h) = -I$ *and* ℓ_j *denote the homotopy class of the core of* S_j *for* $j = 0, \dots, m$*. Then* $\rho(\ell_j)$ *is of even order for all* j*.*

Proof. Set $\alpha_0 = 1$ and $\beta_0 = -b$, which are the corresponding slope to the solid torus S_0. We can express each ℓ_j ($j = 0, 1, \dots, m$) as $\ell_j = q_j^{\mu_j} h^{\nu_j}$ where integers μ_j and ν_j satisfy that $\alpha_j \nu_j - \beta_j \mu_j = -1$ and $0 < \mu_j < \alpha_j$. We will show that every $\rho(\ell_j)^{\alpha_j}$ turns into $-I$. For $j = 0$, the matrix $\rho(\ell_0)^{\alpha_0} (= \rho(\ell_0))$ turns out

$$\begin{aligned}
\rho(\ell_0)^{\alpha_0} &= \rho(q_1 \cdots q_m [a_1, b_1] \cdots [a_g, b_g])^{\alpha_0 \mu_0} \rho(h)^{\alpha_0 \nu_0} \\
&= \rho(h)^{b\mu_0 + \nu_0} \\
&= -I
\end{aligned}$$

Similarly $\rho(\ell_j)^{\alpha_j}$ turns into

$$\rho(q_j)^{\alpha_j \mu_j} \rho(h)^{\alpha_j \nu_j} = \rho(h)^{\alpha_j \nu_j - \beta_j \mu_j} = -I.$$

Hence, for all j, the eigenvalues of $\rho(\ell_j)$ is given by $e^{\pm \pi \eta_j \sqrt{-1}/\alpha_j}$ where some odd integer η_j, which implies that $\rho(\ell)$ is of even order. □

Remark 4.3. For every $j = 0, 1, \dots, m$, it holds that $\rho(\ell_j)^{\alpha_j} = -I$ and $\rho(\ell_j)^{2\alpha_j} = I$. The order of $\rho(\ell_j)$ must be less than or equal to $2\alpha_j$. However the order of $\rho(\ell_0)$ is always 2.

Now we are in position to apply our surgery formula (Theorem 3.6) to a Seifert fibered space $M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$.

Theorem 4.4. *Let ρ be an $\mathrm{SL}_2(\mathbb{C})$-representation of $M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$ such that $\rho(h) = -I$. Then we can express the asymptotics of $\log|\mathrm{Tor}(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|$ as follows:*

$$\text{(i)}\ \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{(2N)^2} = 0,$$

$$\text{(ii)}\ \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{2N} = -\Big(2 - 2g - \sum_{j=1}^{m} \frac{\lambda_j - 1}{\lambda_j}\Big)\log 2$$

where $2\lambda_j$ is the order of $\rho(\ell_j)$.

In particular, if λ_j is equal to α_j for all j, then we have

$$\begin{aligned}\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{2N} &= -\Big(2 - 2g - \sum_{j=1}^{m} \frac{\alpha_j - 1}{\alpha_j}\Big)\log 2 \\ &= -\chi \log 2\end{aligned}$$

where χ is the Euler characteristic of the base orbifold.

We turn to the higher dimensional Reidemeister torsion of M for ρ_{2N}. We have the following explicit values from the assumption that $\rho(h) = -I$.

Proposition 4.5 (Proposition 3.1 in [Kit96])**.** *The Reidemeister torsion $\mathrm{Tor}(M; \rho_{2N})$ is given by $2^{-2N(1-2g-m)}$, i.e., $\log|\mathrm{Tor}(M; \rho_{2N})| = -2N(1 - 2g - m)\log 2$.*

Proposition 4.5 and Theorem 3.6 deduce Theorem 4.4.

Proof of Theorem 4.4. Applying Theorem 3.6, we obtain

$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{(2N)^2} = \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M; \rho_{2N})|}{(2N)^2} = 0$$

by Proposition 4.5. Also it follows that

$$\begin{aligned}\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M(\frac{1}{-b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{2N} &= \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M; \rho_{2N})|}{2N} - \Big(\sum_{j=0}^{m} \frac{1}{\lambda_j}\Big)\log 2 \\ &= -(1 - 2g - m) - \Big(1 + \sum_{j=1}^{m} \frac{1}{\lambda_j}\Big)\log 2 \\ &= -\Big(2 - 2g - \sum_{j=1}^{m} \frac{\lambda_j - 1}{\lambda_j}\Big)\log 2.\end{aligned}$$

∎

Remark 4.6. It follows from the proof of Lemma 4.2 that $\rho(\ell_j)^{2\alpha_j} = I$ for all j. Each λ_j in Theorem 4.4 divides the corresponding α_j.

Corollary 4.7. *The value $-\chi \log 2$ is the maximum in the limits* (ii) *of Theorem 4.4 for all* $\mathrm{SL}_2(\mathbb{C})$*-representations sending h to $-I$.*

Proof. We can rewrite (ii) in Theorem 4.4 as

$$\lim_{N\to\infty} \frac{\mathrm{Tor}(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m});\rho_{2N})}{2N} = -\Big(2-\sum_{j=1}^{m}\frac{\lambda_j-1}{\lambda_j}\Big)\log 2$$

$$= -\chi\log 2 - \log 2\sum_{j=1}^{m}\Big(\frac{1}{\lambda_j}-\frac{1}{\alpha_j}\Big). \tag{13}$$

Our claim follows from that each λ_j is a divisor of α_j for $j=1,\dots,m$. □

We also give the explicit form of higher dimensional Reidemeister torsion for a Seifert fibered space. The following is the direct application of Lemma 2.3 (Multiplicativity Lemma).

Proposition 4.8. *Let ρ be an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M(\frac{1}{-b},\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m}))$ such that $\rho(h)=-I$. Then we can express $\mathrm{Tor}(M(\frac{1}{-b},\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m});\rho_{2N})$ as*

$$\mathrm{Tor}(M(\tfrac{1}{-b},\tfrac{\alpha_1}{\beta_1},\dots,\tfrac{\alpha_m}{\beta_m});\rho_{2N}) = 2^{-2N(2-2g-m)}\cdot\prod_{j=1}^{m}\prod_{k=1}^{N}\Big(2\sin\frac{\pi(2k-1)\eta_j}{2\alpha_j}\Big)^{-2}$$

where $e^{\pm\pi\eta_j\sqrt{-1}/\alpha_j}$ is the eigenvalues of $\rho(\ell_j)$.

For the Reidemeister torsion of Seifert fibered spaces ($g>1$) with irreducible $\mathrm{SL}_n(\mathbb{C})$-representations, we refer to [Kit96].

Remark 4.9. We do not require the irreducibility of $\rho_{2N}=\sigma_n\circ\rho$. However our assumption that $\rho(h)=-I$ guarantee the acyclicity of ρ_{2N} for all N.

4.3. **The leading coefficients and the $\mathrm{SU}(2)$-character varieties for Seifert fibered homology spheres.** We have shown the explicit values of limit of the leading coefficients in higher dimensional Reidemeister torsions for Seifert fibered spaces $M(\frac{1}{-b},\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m})$. The limit of the leading coefficient depends only on the order of $\rho(\ell_j)$ for an $\mathrm{SL}_2(\mathbb{C})$-representation ρ of $\pi_1(M(\frac{1}{-b},\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m}))$. In the fundamental group $\pi_1(M(\frac{1}{-b},\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m}))$, we have the relations that $\ell_j=q_j^{\mu_j}h^{\nu_j}$ and $q_j^{\alpha_j}h^{\beta_j}=1$ where $\alpha_j\nu_j-\beta_j\nu_j=-1$. Under the assumption that $\rho(h)=-I$, the order of $\rho(\ell_j)$ is determined by the order $\rho(q_j)$, i.e., the eigenvalues of $\rho(q_j)$.

Here and subsequently, we assume that ρ is irreducible, following the previous studies [FS90, KK91, BO90]. Irreducible representations with the same eigenvalues for the generators $q_1,\dots,q_m$ form a set with a structure of variety. When we also consider their conjugacy classes, it is known that the set of conjugacy classes has a structure of variety.

We focus on Seifert fibered homology spheres and SU(2)-representations of their fundamental groups. It has shown by [FS90, KK91, BO90] that the set of conjugacy classes of irreducible SU(2)-representations for a Seifert fibered homology sphere can be regarded as the set of smooth manifolds with even dimensions. In the remain of paper, we deal with Seifert fibered homology spheres $M(\frac{1}{-b},\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m})$ with $b=0$. We will denote it briefly by $M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m})$ and write the set of conjugacy classes of irreducible SU(2)-representations of $\pi_1(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m}))$ as

$$\mathcal{R}(M(\tfrac{\alpha_1}{\beta_1},\dots,\tfrac{\alpha_m}{\beta_m})) = \mathrm{Hom}^{\mathrm{irr}}(\pi_1(M(\tfrac{\alpha_1}{\beta_1},\dots,\tfrac{\alpha_m}{\beta_m})),\mathrm{SU}(2))\,/\,\mathrm{conj}.$$

which is called *the* $\mathrm{SU}(2)$*-character variety*.

Each component in $\mathcal{R}(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m}))$ determined by the set of eigenvalues for SU(2)-elements corresponding to $q_1,\dots,q_m$. By the relation that $q_j^{\alpha_j}h^{\beta_j} = 1$, the eigenvalues of $\rho(q_j)$ for an irreducible SU(2)-representation ρ are given by $e^{\pm\pi\xi_j\sqrt{-1}/\alpha_j}$ $(0 \leq \xi_j \leq \alpha_j)$. We will use the sequence $(\xi_1,\dots,\xi_m)$ to denote the corresponding component in $\mathcal{R}(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m}))$ (for details, we refer to [FS90, KK91, BO90]).

Proposition 4.10 ([FS90, KK91, BO90])**.** *Let ρ be an irreducible* SU(2)*-representation of $M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m})$. Suppose that the conjugacy class of ρ is contained in a component $(\xi_1,\dots,\xi_m)$. Then the dimension of $(\xi_1,\dots,\xi_m)$ is equal to $2(n-3)$ where n is the number of ξ_j such that $\xi_j \neq 0, \alpha_j$, i.e.,$\rho(q_j) \neq \pm I$, in $j = 1,\dots,m$.*

We consider which components of $\mathcal{R}(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m}))$ contain the conjugacy class of an irreducible SU(2)-representation ρ when the leading coefficient of $\log|\operatorname{Tor}(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m});\rho_{2N})|$ converge to $-\chi\log 2$.

Proposition 4.11. *Let ρ be an irreducible* SU(2)*-representation such that $\rho(h) = -I$. The leading coefficient $\log|\operatorname{Tor}(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m});\rho_{2N})|/(2N)$ converges to $-\chi\log 2$ if and only if the conjugacy class of ρ is contained in a $2(m-3)$-dimensional component $(\xi_1,\dots,\xi_m)$ in $\mathcal{R}(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m}))$ such that α_j and ξ_j are coprime for all j.*

Before proving Proposition 4.11, let us observe a relation between the order of $\rho(\ell_j)$ and ξ_j.

Lemma 4.12. *Suppose that an irreducible* SU(2)*-representation ρ satisfies that $\rho(h) = -I$ and its conjugacy class is contained in $(\xi_1,\dots,\xi_m)$. Then the order of $\rho(\ell_j)$ is equal to $2\alpha_j$, i.e., $\lambda_j = \alpha_j$, if and only if ξ_j and α_j are coprime.*

Proof. We can express ℓ_j as $\ell_j = q_j^{\mu_j}h^{\nu_j}$ where $\alpha_j\nu_j - \beta_j\mu_j = -1$. Since the eigenvalues of $\rho(q_j)$ are given by $e^{\pm\pi\xi_j\sqrt{-1}/\alpha_j}$, we can diagonalize $\rho(\ell_j)$ as

$$\rho(\ell_j) \sim \begin{pmatrix} e^{\pi\eta_j\sqrt{-1}/\alpha_j} & 0 \\ 0 & e^{-\pi\eta_j\sqrt{-1}/\alpha_j} \end{pmatrix} \quad (\eta_j = \mu_j\xi_j - \alpha_j\nu_j).$$

Suppose that ξ_j and α_j are coprime. It follows from $(\alpha_j,\mu_j) = 1$ that the g.c.d. (α_j,η_j) coincides with $(\alpha,\xi_j) = 1$. Since the order of $\rho(\ell_j)$ is $2\lambda_j$, we can see that $\rho(\ell_j)^{\lambda_j} = -I$. Hence α_j divides λ_j, which implies that $\lambda_j = \alpha_j$ from that λ_j is a divisor of α_j. Similarly, when the order of $\rho(\ell_j)$ is $2\alpha_j$, the g.c.d. (α_j,ξ_j) must be 1. □

Remark 4.13. Under the assumption that $\rho(h) = -I$, if ξ_j is equal to 0 or α_j, then $\rho(\ell_j) = -I$. Hence $\lambda_j = 1$.

Proof of Proposition 4.11. According to Eq. (13), every λ_j coincides with α_j for all j if and only if the leading coefficient of $\log|\operatorname{Tor}(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m});\rho_{2N})|$ converges to $-\chi\log 2$. By Lemma 4.12, we can rephrase $\lambda_j = \alpha_j$ for all j as $[\rho] \in (\xi_1,\dots,\xi_m)$ with the g.c.d. $(\alpha_j,\xi_j) = 1$ for all j. In particular, it is seen from Proposition 4.10 that the dimension of $(\xi_1,\dots,\xi_m)$ equals $2(m-3)$. □

In special cases that every α_j is prime in the Seifert index of $M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m})$, we obtain a simple correspondence between the limits of $\operatorname{Tor}(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m});\rho_{2N})$ and components of $\mathcal{R}(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m}))$.

Theorem 4.14. *Let ρ be an irreducible* SU(2)*-representation of $\pi_1(M(\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_m}{\beta_m}))$ such that $\rho(h) = -I$. Suppose that every α_j is prime and $\alpha_1 < \cdots < \alpha_m$. If the conjugacy*

class of ρ *is contained in* $(\xi_1, \ldots, \xi_m)$*, then the limit of* $\log|\operatorname{Tor}(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|/(2N)$ *expressed as*

$$\lim_{N\to\infty} \frac{\log|\operatorname{Tor}(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{2N} = -\Big(2 - \sum_{\xi_j \neq 0, \alpha_j} \frac{\alpha_j - 1}{\alpha_j}\Big) \log 2. \tag{14}$$

If the set $\{[\rho] \in \mathcal{R}(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})) \,|\, \rho(h) = -I\}$ *has* 0 *and* $2(m-3)$*-dimensional components, then the limit takes the maximum* $-\chi \log 2$ *on only all top-dimensional components and takes the minimum on some* 0*-dimensional components.*

Proof. As seen in Remark 4.6, we have $\lambda_j = 1$ if $\xi_j = 0$ or α_j. Substituting $\lambda_j = 1$ into Eq. (13) for the corresponding index j, we have the limit (14). It follows from Proposition 4.11 and our assumption that the limit takes the maximum $-\chi \log 2$ on all top-dimensional components.

It remains to prove the limit takes the minimum on a 0-dimensional component. Since the limit is expressed as Eq. (14), we consider the minimum of $\sum_{\xi_j \neq 0, \alpha_j} (\alpha_j - 1)/\alpha_j$. A 0-dimensional component is given by $(\xi_1, \ldots, \xi_m)$ for all $\xi_j = 0, \alpha_j$ except three ξ_{j_1}, ξ_{j_2} and ξ_{j_3}. We need to consider two cases: (i) $\alpha_1 = 2$ and (ii) $\alpha_1 \geq 3$. In the case that $\alpha_1 = 2$, ξ_1 must be 1 for all components $(\xi_1, \ldots, \xi_m)$ since we have $\rho(q_1)^{\alpha_1} = -I$ from the assumption that $\rho(h) = -I$. The sum $(\alpha_{j_1} - 1)/\alpha_{j_1} + (\alpha_{j_2} - 1)/\alpha_{j_2} + (\alpha_{j_3} - 1)/\alpha_{j_3}$ turns into

$$\frac{1}{2} + \frac{\alpha_{j_2} - 1}{\alpha_{j_2}} + \frac{\alpha_{j_3} - 1}{\alpha_{j_3}} < \frac{5}{2}.$$

On the other hand, it is easily seen that for higher dimensional components,

$$\frac{1}{2} + \frac{\alpha_{i_2} - 1}{\alpha_{i_2}} + \frac{\alpha_{i_3} - 1}{\alpha_{i_3}} + \frac{\alpha_{i_4} - 1}{\alpha_{i_4}} + \cdots \geq \frac{1}{2} + \frac{3-1}{3} + \frac{5-1}{5} + \frac{7-1}{7} > \frac{5}{2}.$$

Hence the minimum lies in a 0-dimensional component.

In the other case that $\alpha_1 \geq 3$, it is clear that

$$\frac{\alpha_{j_1} - 1}{\alpha_{j_1}} + \frac{\alpha_{j_2} - 1}{\alpha_{j_2}} + \frac{\alpha_{j_3} - 1}{\alpha_{j_3}} < 3.$$

On the other hand, we can see that

$$\frac{\alpha_{i_1} - 1}{\alpha_{i_1}} + \frac{\alpha_{i_2} - 1}{\alpha_{i_2}} + \frac{\alpha_{i_3} - 1}{\alpha_{i_3}} + \frac{\alpha_{i_4} - 1}{\alpha_{i_4}} + \cdots \geq \frac{3-1}{3} + \frac{5-1}{5} + \frac{7-1}{7} + \frac{11-1}{11} > 3.$$

The minimum of the limit lies on 0-dimensional components. □

4.4. **Examples for Seifert fibered homology spheres.** We will see two examples of Theorem 4.14 and an example which shows that $\log|\operatorname{Tor}(M(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|/(2N)$ does not converges to the maximum $-\chi \log 2$ on all top–dimensional components if some α_j is not prime.

4.4.1. $M(\frac{2}{\beta_1}, \frac{3}{\beta_2}, \frac{7}{\beta_3})$. We can choose that $\beta_1 = 1$ and $\beta_2 = \beta_3 = -1$ by the requirement that $2 \cdot 3 \cdot 7(\beta_1/2 + \beta_2/3 + \beta_3/7) = 1$. The Brieskorn homology 3-sphere $M(\frac{2}{1}, \frac{3}{-1}, \frac{7}{-1})$ also corresponds to the surgery along $(2,3)$-torus knot with slope 1 in Subsection 3.3. From the presentation:

$$\pi_1(M(\tfrac{2}{1}, \tfrac{3}{-1}, \tfrac{7}{-1})) = \langle q_1, q_2, q_3, h \,|\, [q_j, h] = 1, q_j^{\alpha_j} h^{\beta_j} = 1, q_1 q_2 q_3 = 1 \rangle,$$

every irreducible SU(2)-representation sends h to $-I$ and the SU(2)-character variety of $M(\frac{2}{1}, \frac{3}{-1}, \frac{7}{-1})$ consists of $(\xi_1, \xi_2, \xi_3) = (1, 1, 3)$ and $(1, 1, 5)$. For details about the computation of SU(2)-character varieties, we refer to [FS90, KK91] and [Sav99, Lecture 14].

Let ρ be an irreducible SU(2)-representation of $M(\frac{2}{1}, \frac{3}{-1}, \frac{7}{-1})$. By the relations that $\ell_j = q_j^{\mu_j} h^{\nu_j}$ and $\alpha_j \nu_j - \beta_j \mu_j = -1$ $(0 < \mu_j < \alpha_j)$, we obtain that

$$\rho(\ell_1) = \rho(q_1),\ \rho(\ell_2) = -\rho(q_2)^2,\ \rho(\ell_3) = -\rho(q_3)^6.$$

From Eqs. $\rho(q_1)^2 = -I$, $\rho(q_2)^3 = -I$ and $\rho(q_3)^7 = -I$, the orders $2\lambda_j$ of $\rho(\ell_j)$ are given by

$$2\lambda_1 = 4,\ 2\lambda_2 = 6,\ 2\lambda_3 = 14.$$

By Theorem 4.4, for the both cases of $[\rho] \in (1, 1, 3)$ and $[\rho] \in (1, 1, 5)$, we can see that

$$\lim_{N\to\infty} \frac{\log|\operatorname{Tor}(M(\frac{2}{1}, \frac{3}{-1}, \frac{7}{-1}); \rho_{2N})|}{2N} = \Big(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{7}\Big) \log 2.$$

The limit of the leading coefficient takes the maximum $-\chi \log 2$ on every top–dimensional components (see also Theorem 3.14).

4.4.2. $M(\frac{2}{\beta_1}, \frac{3}{\beta_2}, \frac{5}{\beta_3}, \frac{7}{\beta_4})$. Let us choose $\beta_1 = 1$, $\beta_2 = \beta_3 = -2$ and $\beta_4 = 4$. The subvariety $\{[\rho] \in \mathcal{R}(M(\frac{2}{1}, \frac{3}{-2}, \frac{5}{-2}, \frac{7}{4})) \,|\, \rho(h) = -I\}$ consists of eight points and six 2-dimensional spheres. (For details, see [Sav99, Lecture 14])

Each 0-dimensional component corresponds to the parameter $(\xi_1, \xi_2, \xi_3, \xi_4)$ in:

$$\left\{ \begin{array}{c} (1,0,2,2),\ (1,0,2,4),\ (1,0,2,6),\ (1,0,4,4), \\ (1,2,0,2),\ (1,2,0,4),\ (1,2,2,0),\ (1,2,4,0) \end{array} \right\} \tag{15}$$

and each 2-dimensional components are given by $(\xi_1, \xi_2, \xi_3, \xi_4)$ in

$$\left\{ \begin{array}{c} (1,2,2,2),\ (1,2,2,4),\ (1,2,2,6), \\ (1,2,4,2),\ (1,2,4,4),\ (1,2,4,6) \end{array} \right\}.$$

We can express ℓ_j $(1 \le j \le 4)$ as

$$\ell_1 = q_1,\ \ell_2 = q_2 h^{-1},\ \ell_3 = q_3^2 h^{-1},\ \ell_4 = q_4 h.$$

Let ρ be an irreducible SU(2)-representation of $\pi_1(M(\frac{2}{1}, \frac{3}{-2}, \frac{5}{-2}, \frac{7}{4}))$ such that $\rho(h) = -I$. We have the following table between the 0-dimensional components and the orders of $\rho(\ell_j)$ for $j = 1, 2, 3, 4$:

$(\xi_1, \xi_2, \xi_3, \xi_4)$	λ_j: the half of the order of $\rho(\ell_j)$
$(\xi_1, 0, \xi_3, \xi_4)$	$\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 5,\ \lambda_4 = 7$
$(\xi_1, \xi_2, 0, \xi_4)$	$\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 1,\ \lambda_4 = 7$
$(\xi_1, \xi_2, \xi_3, 0)$	$\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 5,\ \lambda_4 = 1.$

Hence for $[\rho] \in (\xi_1, \xi_2, \xi_3, \xi_4)$ $(\xi_2 = 0)$, by Theorem 4.4, we obtain

$$\begin{aligned} \lim_{N\to\infty} \frac{\log|\operatorname{Tor}(M(\frac{2}{1}, \frac{3}{-2}, \frac{5}{-2}, \frac{7}{4}); \rho_{2N})|}{2N} &= -\Big(2 - \sum_{j\neq 2} \frac{\lambda_j - 1}{\lambda_j}\Big) \log 2 \\ &= -\chi \log 2 - \frac{2}{3} \log 2, \end{aligned}$$

for $[\rho] \in (\xi_1, \xi_2, \xi_3, \xi_4)$ $(\xi_3 = 0)$,

$$\begin{aligned} \lim_{N\to\infty} \frac{\log|\operatorname{Tor}(M(\frac{2}{1}, \frac{3}{-2}, \frac{5}{-2}, \frac{7}{4}); \rho_{2N})|}{2N} &= -\Big(2 - \sum_{j\neq 3} \frac{\lambda_j - 1}{\lambda_j}\Big) \log 2 \\ &= -\chi \log 2 - \frac{4}{5} \log 2 \end{aligned}$$

and for $[\rho] \in (\xi_1, \xi_2, \xi_3, \xi_4)$ $(\xi_4 = 0)$,

$$\lim_{N\to\infty} \frac{\log|\operatorname{Tor}(M(\frac{2}{1}, \frac{3}{-2}, \frac{5}{-2}, \frac{7}{4}); \rho_{2N})|}{2N} = -\Big(2 - \sum_{j\neq 4} \frac{\lambda_j - 1}{\lambda_j}\Big)\log 2$$
$$= -\chi \log 2 - \frac{6}{7}\log 2.$$

When the conjugacy class $[\rho]$ is contained in the 2-dimensional components in (15), then it is seen that $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5$ and $\lambda_4 = 7$. Hence we obtain the maximal value of the limit:

$$\lim_{N\to\infty} \frac{\log|\operatorname{Tor}(M(\frac{2}{1}, \frac{3}{-2}, \frac{5}{-2}, \frac{7}{4}); \rho_{2N})|}{2N} = -\chi \log 2.$$

The limit of $\log|\operatorname{Tor}(M(\frac{2}{1}, \frac{3}{-2}, \frac{5}{-2}, \frac{7}{4}); \rho_{2N})|/(2N)$ takes the minimum at the components $(1, 2, 2, 0)$ and $(1, 2, 4, 0)$.

4.4.3. $M(\frac{5}{\beta_1}, \frac{6}{\beta_2}, \frac{7}{\beta_3})$. Let us choose $\beta_1 = 3$, $\beta_2 = -1$ and $\beta_3 = -3$. The subvariety $\{[\rho] \in \mathcal{R}(M(\frac{5}{3}, \frac{6}{-1}, \frac{7}{-3})) \mid \rho(h) = -I\}$ consists of eight points which are given by the following set of (ξ_1, ξ_2, ξ_3):

$$\left\{ \begin{array}{c} (1,1,1),\ (1,3,3),\ (1,5,5), \\ (3,1,5),\ (3,3,1),\ (3,3,3),\ (3,3,5),\ (3,5,3) \end{array} \right\}$$

We separate the above subvariety into two subset X_1 and X_2 as follows:

$$X_1 := \{(1,1,1), (1,5,5), (3,1,5), (3,5,3)\}$$
$$X_2 := \{(1,3,3), (3,3,1), (3,3,3), (3,3,5)\}$$

where every component is 0-dimensional. Every components in X_1 satisfies that $(\alpha_j, \xi_j) = 1$ for all j. On the other hand, each components in X_2 satisfies that $(\alpha_1, \xi_1) = (\alpha_3, \xi_3) = 1$ and $(\alpha_2, \xi_2) = 3$.

In the case that the conjugacy class $[\rho]$ is contained in X_1, we can also compute similarly $\lambda_1 = 5$, $\lambda_2 = 6$ and $\lambda_3 = 7$. Therefore we obtain, by Theorem 4.4

$$\lim_{N\to\infty} \frac{\log|\operatorname{Tor}(M(\frac{5}{3}, \frac{6}{-1}, \frac{7}{-3}); \rho_{2N})|}{2N} = -\chi \log 2.$$

In the case of the conjugacy class $[\rho]$ is contained in X_2, the order of $\rho(\ell_2)$ is 4 since $\ell_2 = q_2^5 h^{-1}$ and $\rho(q_2)^2 = -I$. Hence λ_2 equals to 2. Similar computations yield $\lambda_1 = 5$ and $\lambda_3 = 7$. Furthermore Theorem 4.4 gives the following limit:

$$\lim_{N\to\infty} \frac{\log|\operatorname{Tor}(M(\frac{5}{3}, \frac{6}{-1}, \frac{7}{-3}); \rho_{2N})|}{2N} = -\Big(2 - \sum_{j=1}^{3} \frac{\lambda_j - 1}{\lambda_j}\Big)\log 2$$
$$= -\chi \log 2 - \frac{1}{3}\log 2.$$

Therefore We have the top-dimensional components where the limit of the leading coefficient in $\log|\operatorname{Tor}(M(\frac{5}{3}, \frac{6}{-1}, \frac{7}{-3}); \rho_{2N})|$ does not take the maximum.

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